Some Contributions to Convex Infinite-Dimensional Optimization Duality

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Introduction

Consider the convex infinite programming problem

\[ (P) \quad \text{Min} \quad f(x) \]
\[ \text{s.t.} \quad f_t(x) \leq 0, \ t \in T, \]
\[ x \in C, \]  \hspace{1cm} (1) \]

where:

- \( T \): index set (possibly infinite)
- \( C \): convex subset of a lcHtv X (\( C \neq \emptyset \))
- \( f; f_t, \ t \in T \): proper convex functions defined on X
- Feasible set of \((P)\): \( F \cap C \), where
  \[ F := \{ x \in X : f_t(x) \leq 0, \ t \in T \} \]
- Optimal value: \( \inf(P) \in [\neg \infty, +\infty] \)
- Optimal set: \( S(P) = \{ x \in F \cap C : f(x) = \inf(P) \} \) (possibly, empty)
We consider two different associated duals:

- **Lagrangian dual of (P):**

  \[
  (D) \quad \text{Max}_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in \mathcal{C}} \{ f(x) + \sum_{t \in T} \lambda_t f_t(x) \}
  \]  

  where

  \[
  \mathbb{R}_{+}^{(T)} := \left\{ \lambda \equiv (\lambda_t)_{t \in T} \in \mathbb{R}_+^T \mid \text{only finitely many } \lambda_t > 0 \right\},
  \]

  and

  \[
  \mathbb{R}^{(T)} \times \mathbb{R}^T \ni \langle \lambda, f \rangle := \sum_{t \in T} \lambda_t f_t(x) := \begin{cases} 
  0, & \text{if } \lambda = 0_T, \\
  \sum_{t \in \text{supp } \lambda} \lambda_t f_t(x), & \text{if } \lambda \neq 0_T.
  \end{cases}
  \]
We consider two different associated duals:

- **Lagrangian dual of** $(P)$:

  $$(D) \quad \max_{\lambda \in \mathbb{R}^T_+} \inf_{x \in C} \{ f(x) + \sum_{t \in T} \lambda_t f_t(x) \} \tag{2}$$

  where

  $$\mathbb{R}^T_+ := \{ \lambda \equiv (\lambda_t)_{t \in T} \in \mathbb{R}^T \mid \text{only finitely many } \lambda_t > 0 \} ,$$

  and

  $$\mathbb{R}^T \times \mathbb{R}^T \ni \langle \lambda, f \rangle := \sum_{t \in T} \lambda_t f_t(x) := \begin{cases} 0, & \text{if } \lambda = 0_T, \\ \sum_{t \in \text{supp } \lambda} \lambda_t f_t(x), & \text{if } \lambda \neq 0_T. \end{cases}$$

- **Modified dual of** $(P)$:

  $$(\Delta) \quad \max_{\lambda \in \mathbb{R}^T_+ \setminus \{0_T\}} \inf_{x \in C} \{ f(x) + \sum_{t \in T} \lambda_t f_t(x) \} \tag{3}$$

- **Weak dual inequalities**:

  $$-\infty \leq \sup(\Delta) \leq \sup(D) \leq \inf(P) \leq +\infty. \tag{4}$$
MAIN GOAL: To establish conditions for the converse strong Lagrangian duality (minsup duality)

\[ \min(P) = \sup(D), \]

and also for strong Lagrangian duality (infmax duality)

\[ \inf(P) = \max(D). \]
MAIN GOAL: To establish conditions for the converse strong Lagrangian duality (minsup duality)

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Outline:

- Preliminaries
- Basic results for Infmax duality
- Minsup duality by assuming inf-compactness
- Minsup duality without inf-compactness
- Applications
Example 1 Let $X = C = \mathbb{R}^2, f(x) = \exp(x_2), T = \{1\},$ and $f_1(x) = x_1 + i_{\mathbb{R} \times \mathbb{R}^+}(x).$ We compute

$$\sup(\Delta) = -\infty < \max(D) = 0 \text{ (attained for } \lambda = 0) < 1 = \min(P)$$

Slater condition does not guarantee even $\inf(P) = \sup(D).$
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Example 2 Let $X = C = \mathbb{R}, f (x) = \exp (x), T = \{1\},$ and $f_1 (x) = x$. Then

$$\max (\Delta) = -\infty < \max (D) = 0 = \inf (P).$$

In this case, Slater condition holds and, however, $\sup (\Delta) \neq \sup (D)$. 

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Example 1 Let \( X = C = \mathbb{R}^2, f(x) = \exp(x_2), T = \{1\}, \) and \( f_1(x) = x_1 + i_{\mathbb{R} \times \mathbb{R}^+}(x) \). We compute

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\max(\Delta) = -\infty < \max(D) = 0 = \inf(P).
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In this case, Slater condition holds and, however, \( \sup(\Delta) \neq \sup(D) \).

Example 3 Let \( X = \mathbb{R}, C = [-1, 1], f(x) = -x, T = \{1\}, \) and \( f_1(x) = x \) if \( x \geq 0, f_1(x) = 0 \) if \( x < 0 \). Now

\[
\max(\Delta) = \max(D) = \min(P) = 0.
\]

However, Slater condition is not satisfied.
Preliminaries

- Given $B \subset X$, we denote by $\text{co } B$, $\text{cone } B$, and $\text{aff } B$ the convex hull of $B$, the smallest convex cone containing $B \cup \{0_X\}$, and the smallest linear manifold containing $B$, respectively.

- The duality product of $x^* \in X^*$ (the topological dual of $X$) and $x \in X$ is represented by $\langle x^*, x \rangle$.

- The *positive and negative dual cones* of a non-empty set $C \subset X$ are
  \[ C^+ = \{ x^* \in X^* : \langle x^*, x \rangle \geq 0 \ \forall x \in C \} , \]
  and
  \[ C^- = \{ x^* \in X^* : \langle x^*, x \rangle \leq 0 \ \forall x \in C \} . \]

- The *recession cone* of a non-empty convex set $C \subset X$ is
  \[ C_\infty = \{ v \in X : c + v \in C \ \forall c \in C \} . \]
Infmax duality

We are dealing with the problem

\[(P) \quad \text{Min} \quad \{f(x), \text{ s.t. } f_t(x) \leq 0, t \in T, x \in C\},\]

assuming that $C$ is closed and $f,f_t \in \Gamma(X)$. The function $h : X^* \to \overline{\mathbb{R}}$

$$h := \inf_{\lambda \in \mathbb{R}^T_+ \setminus \{0_T\}} \left( f + i_C + \sum_{t \in T} \lambda_t f_t \right)^*,$$

is crucial in our approach; $h$ enjoys the following properties:
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(i) \( h \) is proper and convex, and \(-h(0_{X^*}) = \sup(\Delta)\)
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(i) \( h \) is proper and convex, and \(-h (0_{X^*}) = \sup (\Delta)\)

(ii) \( h^* = f + i_{\mathbb{C} \cap F} \)
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(ii) \( -h^{**}(0_{X^*}) = \inf (P) \in \overline{\mathbb{R}} \)
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is crucial in our approach; \( h \) enjoys the following properties:

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(ii) \( -h^{**}(0_{X^*}) = \inf(P) \in \overline{\mathbb{R}} \)

\( \Gamma(X^*): w^*\)-lsc proper convex functions.
Let us introduce the set

\[ A := \bigcup_{\lambda \in \mathbb{R}_+^{(T)} \setminus \{0_T\}} \operatorname{epi} \left( f + i_C + \sum_{t \in T} \lambda_t f_t \right)^* \]
Let us introduce the set

\[ \mathcal{A} := \bigcup_{\lambda \in \mathbb{R}_+^{(T)} \setminus \{0_T\}} \text{epi} \left( f + i_C + \sum_{t \in T} \lambda_t f_t \right)^* \]

It holds that

\[ \text{epi}_s h \subset \mathcal{A} \subset \text{epi} h, \quad (5) \]
\[ \text{epi} \overline{h} = \text{cl}^{\mu*} \mathcal{A}, \quad (6) \]

and

\[ \overline{h} = (f + i_{C \cap M})^* = h^{**}. \quad (7) \]
Definition ([2])

Having two subsets $A$ and $B$ of a topological space $(X, \tau)$, $A$ is said to be \textit{closed regarding to} $B$ if $B \cap \text{cl} A = B \cap A$. 

Remark
Closedness of $A$ regarding $B$ is clearly stronger than the closedness of $A$ in the topology induced by $\tau$ in $B$, as the last one only requires that $B \setminus A$ be the intersection of a closed set in $(X, \tau)$ with $B$ (not specifically $\text{cl} A$).
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We are now in a position to state the main result for infmax duality.
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We are now in a position to state the main result for infmax duality.

**Theorem**

Assume that $\inf (P) < +\infty$. The following assertions are equivalent:

(i) $A$ is $w^*$-closed regarding to the set $\{0_{X^*}\} \times \mathbb{R}$.

(ii) $\inf (P) = \max(\Delta)$, including the value $-\infty$.
Minsup duality under inf-compactness

Theorem

\[ \bar{x} \in \partial h(0_{X^*}) \iff \bar{x} \in S(P) \text{ and } \min(P) = \sup(\Delta). \]
Minsup duality under inf-compactness

**Theorem**

\[
\bar{x} \in \partial h \left( 0_{X^*} \right) \iff \bar{x} \in S(P) \text{ and } \min(P) = \sup(\Delta).
\]

- \( g : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{ \pm \infty \} \) is said to be *inf-compact* (inf-locally-compact) when \([g \leq r] := \{ x \in X : g(x) \leq r \}\) is a compact set (a locally compact set, respectively) for every \( r \in \mathbb{R} \).
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- Additionally

\[ \partial h (0_{X^*}) \neq \emptyset \iff h \text{ is finite and } \tau (X^*, X) \text{-cont. at } 0_{X^*} \]

\[ \iff \begin{array}{c} h \in \Gamma (X^*) \ \text{and} \ \hbar^* \text{ is w-inf-compact,} \\
\iff \end{array} \]

i.e., the sublevel sets of \( h^* \) are \( \sigma (X, X^*) \)-compact.
We are now in position to state our first result.

Theorem

Assume that one of the following conditions holds:

(a) \( \lambda R(T) + \) such that the function \( f + \sum t_2 T \lambda t f t \) is w-inf-compact.

(b) \( x_2 \text{cone}(t_2 T \text{dom} f t) \) such that \( f + x \) is w-inf-compact.

Then \( \min(P) = \sup(D) \) and \( S(P) \) is a non-empty weakly compact set.

For the linear SIP \( (X = \mathbb{R}^n) \) problem \( \text{(P)} \) Min \( fh c, x i, s.t. h a t, x i b, t_2 T, x_2 C g, \) with \( c, a t_2 \mathbb{R}^n, b t_2 \mathbb{R}^n \), the theorem (under \( (b) \)) asserts that, if there exists a \( \text{cone} f a t, t_2 T g \) such that \( C_\infty \{c + a] = f_0 \) \( \mathbb{R}^n g \), then \( \min(P) = \sup(D) \) holds and \( S(P) \) is compact.

If \( C = \mathbb{R}^n \) corollary does not apply since \( \{c + a] \) is either a halfspace or \( \mathbb{R}^n \). So, inf-compactness is useless in linear SIP.
We are now in position to state our first result.

**Theorem**

Assume that one of the following conditions holds:

(a) \( \exists \lambda \in \mathbb{R}_+^T \) such that the function \( f + i_C + \sum_{t \in T} \lambda_t f_t \) is w-inf-compact.

(b) \( \bar{x}^* \in \text{cone}(\bigcup_{t \in T} \text{dom} f_t^*) \) such that \( f + i_C + \bar{x}^* \) is w-inf-compact.
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**Theorem**

Assume that one of the following conditions holds:

(a) $\exists \bar{\lambda} \in \mathbb{R}^T_+$ such that the function $f + i_C + \sum_{t\in T} \bar{\lambda}_t f_t$ is $w$-inf-compact.

(b) $\bar{x}^* \in \text{cone}(\bigcup_{t\in T} \text{dom} f_t^*)$ such that $f + i_C + \bar{x}^*$ is $w$-inf-compact.

Then $\min(P) = \sup(D)$ and $S(P)$ is a non-empty weakly compact set.
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**Theorem**

Assume that one of the following conditions holds:

(a) \( \exists \bar{\lambda} \in \mathbb{R}^{(T)}_+ \) such that the function \( f + \mathbf{i}_C + \sum_{t \in T} \bar{\lambda}_t f_t \) is \( w \)-inf-compact.

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Then \( \min(P) = \sup(D) \) and \( S(P) \) is a non-empty weakly compact set.

For the linear SIP (\( X = X^* = \mathbb{R}^n \)) problem

\[
(P) \quad \text{Min} \quad \{ \langle c^*, x \rangle , \text{ s.t. } \langle a_t^*, x \rangle \leq b, \ t \in T, \ x \in C \},
\]

with \( c^*, a_t^* \in \mathbb{R}^n, b_t \in \mathbb{R} \), the theorem (under (b)) asserts that, if there exists \( a^* \in \text{cone} \{ a_t^*, t \in T \} \) such that \( C_\infty \cap [c^* + a^* \leq 0] = \{0_{\mathbb{R}^n}\} \), then \( \min(P) = \sup(D) \) holds and \( S(P) \) is compact.
We are now in position to state our first result.

**Theorem**

Assume that one of the following conditions holds:

(a) \( \exists \bar{\lambda} \in \mathbb{R}^{(T)}_+ \) such that the function \( f + i_C + \sum_{t \in T} \bar{\lambda}_t f_t \) is \( w \)-inf-compact.

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Then \( \min(P) = \sup(D) \) and \( S(P) \) is a non-empty weakly compact set.

For the linear SIP \( (X = X^* = \mathbb{R}^n) \) problem

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(P) \quad \text{Min} \quad \{ \langle c^*, x \rangle, \text{ s.t. } \langle a^*_t, x \rangle \leq b, \ t \in T, \ x \in C \},
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If \( C = \mathbb{R}^n \) corollary does not apply since \( [c^* + a^* \leq 0] \) is either a halfspace or \( \mathbb{R}^n \). So, inf-compactness is useless in linear SIP.

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Minsup duality without inf-compactness

Definition The function $h \in \Gamma(Y)$, with $Y$ being a l.c.H.t.v.s., is quasicontinuous (Def. 7.6.2 in Laurent’72) if:

a) aff dom $h$ is closed and of finite codimension;

b) ri dom $h \neq \emptyset$ and the restriction of $h$ to aff dom $h$ is continuous on ri dom $h$.

Lemma

Let $h : Y \to \mathbb{R}$ be convex and quasicontinuous, and let $y_0 \in Y$ be such that $h(y_0) > \infty$ and cl cone (dom $h(y_0)$) is a linear subspace. Then $\partial h(y_0)$ is the sum of a non-empty w-compact convex set and a finite dimensional linear subspace.

If cl cone (dom $h(y_0)$) = $Y$, $h$ is continuous at $y_0$, and $\partial h(y_0)$ is a non-empty w-compact convex set.
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**Lemma**

If $g \in \Gamma(X)$, then $g$ is w-inf-locally-compact if and only if $g^*$ is $\tau(X^*, X)$-quasicontinuous.
Minsup duality without inf-compactness

**Definition** The function \( h \in \Gamma(Y) \), with \( Y \) being a l.c.H.t.v.s., is **quasicontinuous** (Def. 7.6.2 in Laurent’72) if:

a) \( \text{aff dom } h \) is closed and of finite codimension;
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**Lemma**

If \( g \in \Gamma(X) \), then \( g \) is \( w \)-inf-locally-compact if and only if \( g^* \) is \( \tau(X^*,X) \)-quasicontinuous.

**Lemma**

Let \( h : Y \to \overline{\mathbb{R}} \) be convex and quasicontinuous, and let \( y_0 \in Y \) be such that \( h(y_0) > -\infty \) and \( \text{cl cone}(\text{dom } h - y_0) \) is a linear subspace. Then \( \partial h(y_0) \) is the sum of a non-empty \( w^* \)-compact convex set and a finite dimensional linear subspace.
Minsup duality without inf-compactness

**Definition** The function \( h \in \Gamma(Y) \), with \( Y \) being a l.c.H.t.v.s., is quasicontinuous (Def. 7.6.2 in Laurent’72) if:

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**Lemma**

If \( g \in \Gamma(X) \), then \( g \) is w-inf-locally-compact if and only if \( g^* \) is \( \tau(X^*, X) \)-quasicontinuous.

**Lemma**

Let \( h : Y \to \overline{\mathbb{R}} \) be convex and quasicontinuous, and let \( y_0 \in Y \) be such that \( h(y_0) > -\infty \) and \( \text{cl cone}(\text{dom } h - y_0) \) is a linear subspace. Then \( \partial h(y_0) \) is the sum of a non-empty \( w^* \)-compact convex set and a finite dimensional linear subspace.

If \( \text{cl cone}(\text{dom } h - y_0) = Y \), \( h \) is continuous at \( y_0 \), and \( \partial h(y_0) \) is a non-empty \( w^* \)-compact convex set.
This is a minsup duality theorem without inf-compactness:

**Theorem**

Assume that \( \sup(\Delta) < +\infty \) and that

\[
\exists \bar{\lambda} \in \mathbb{R}^T_+ \setminus \{0_T\} : f + i_C + \sum_{t \in T} \bar{\lambda}_t f_t \text{ is } w\text{-inf-locally-compact},
\]

holds and that

\[
[f_\infty \leq 0] \cap C_\infty \cap \left( \bigcap_{t \in T} [f_t)_\infty \leq 0] \right) \text{ is a linear subspace, } (8)
\]

where \([f_\infty \leq 0]\) and \([f_t)_\infty \leq 0]\) are the recession cones of \(f\) and \(f_t\).

Then

\[
\min(P) = \sup(D),
\]

and \(S(P)\) is the sum of a non-empty \(w\)-compact convex set and a finite dimensional linear subspace.
Consider now the *(primal)* ordinary convex program

\[(P) \quad \text{Min} \quad \{f(x), \text{ s.t. } f_i(x) \leq 0, \ i = 1, \ldots, m, \ x \in C \subset \mathbb{R}^n\},\]

and its *dual*

\[(D) \quad \text{Max}_{\lambda \in \mathbb{R}^m_+} \ \inf_{x \in C} \left( f(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \right).\]
Consider now the (primal) ordinary convex program

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(P) \quad \text{Min} \quad \{ f(x), \text{ s.t. } f_i(x) \leq 0, \ i = 1, \ldots, m, \ x \in C \subset \mathbb{R}^n \},
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\]

Applying the last theorem we get a very general form of Clark-Duffin Theorem (Duffin'78):

\[
\text{Corollary (Generalized Clark-Duffin Theorem)}
\]

Let \( f, f_1, \ldots, f_m \) \( \in \Gamma(\mathbb{R}^n) \) and \( C \) convex closed in \( \mathbb{R}^n \) be such that \( f_\infty 0 \ \subset C \subset f_\infty 0 \) \( \Rightarrow \) is a linear subspace. Then \( \min (P) = \sup (D) \) and \( S(P) \) is the sum of a non-empty compact convex set and a linear subspace.
Consider now the (primal) ordinary convex program

\[(P) \quad \text{Min} \quad \{f(x) , \text{s.t. } f_i(x) \leq 0, \ i = 1,...,m, \ x \in C \subset \mathbb{R}^n \}\]

and its dual

\[(D) \quad \text{Max}_{\lambda \in \mathbb{R}^m_+} \inf_{x \in C} \left( f(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \right).\]

Applying the last theorem we get a very general form of Clark-Duffin Theorem (Duffin’78):

**Corollary**

*(Generalized Clark-Duffin Theorem)* Let \(f,f_1,...,f_m \in \Gamma (\mathbb{R}^n)\) and \(C \) convex closed in \(\mathbb{R}^n\) be such that

\[
[f_\infty \leq 0] \cap C_\infty \cap \left( \bigcap_{i=1,...,m} [(f_i)_\infty \leq 0] \right)
\]

is a linear subspace. Then \(\min(P) = \sup(D) \in \mathbb{R}\) and \(S(P)\) is the sum of a non-empty compact convex set and a linear subspace.
If we deal with the linear SIP problem

\[(P) \quad \text{Min} \quad \{ \langle c^* , x \rangle , \text{s.t.} \; \langle a_t^* , x \rangle \leq b , \; t \in T \} , \]

with \( X = C = \mathbb{R}^n = X^* , \)

\[ [c^* \leq 0] \cap \left( \bigcap_{t \in T} [a_t^* \leq 0] \right) \]

is a linear subspace

if and only if

\[ -c^* \in \text{ri cone}\{a_t^* , \; t \in T \} . \]
If we deal with the linear SIP problem

\[(P) \quad \text{Min} \quad \{ \langle c^*, x \rangle, \text{ s.t.} \; \langle a^*_t, x \rangle \leq b, \; t \in T \}, \]

with \( X = C = \mathbb{R}^n = X^* \),

\[[c^* \leq 0] \cap \left( \bigcap_{t \in T} [a^*_t \leq 0] \right) \text{ is a linear subspace if and only if} \]

\[-c^* \in \text{ri cone}\{a^*_t, \; t \in T\}.\]

This condition guarantees \( \min(P) = \sup(D) \), and that \( S(P) \) is the sum of a non-empty compact convex set and a linear subspace.
If we deal with the linear SIP problem

\[(P) \quad \operatorname{Min} \{ \langle c^*, x \rangle, \text{ s.t. } \langle a_t^*, x \rangle \leq b, \ t \in T \}, \]

with \( X = C = \mathbb{R}^n = X^* \),

\[ [c^* \leq 0] \cap \left( \bigcap_{t \in T} [a_t^* \leq 0] \right) \] is a linear subspace

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\[-c^* \in \text{ri cone} \{ a_t^*, \ t \in T \}. \]

This condition guarantees \( \min(P) = \sup(D) \), and that \( S(P) \) is the sum of a non-empty compact convex set and a linear subspace.

If, additionally, \( \text{cone} \{ a_t^*, \ t \in T \} \) is full-dimensional, \( S(P) \) is a non-empty compact convex set.
**Definition**

A family \((C_t)_{t \in T}\) of sets of a topological space is said to have the **finite intersection property** if every **finite subfamily** has non-empty intersection.

**Proposition**

Let \((C_t)_{t \in T}\) be a family of closed convex subsets of a lcHtvs having the finite intersection property. Moreover, assume the existence of \(t_1, \ldots, t_m \in T\) such that \(\bigcap_{i=1}^m C_{t_i}\) is \(w\)-locally-compact and \(\bigcap_{t \in T} (C_t)_\infty\) is a linear space. Then \(\bigcap_{t \in T} C_t\) is the sum of a non-empty \(w\)-compact convex set and a finite dimensional linear space.
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Proof: If we apply Theorem 7 with \(C = X, f \equiv 0,\) and \(f_t = \text{i}_{C_t},\) \(t \in T,\) we observe the following:

- \(S(P) = \bigcap_{t \in T} C_t,\)
- \(\text{rec}(P) = \bigcap_{t \in T} (C_t)_\infty,\) and
- \(\text{sup}(\Delta) < +\infty\) amounts to say that the family \((C_t)_{t \in T}\) has the finite intersection property.
REFERENCES


