Noncooperative Games, Couplings Constraints, and Partial Efficiency

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King’s College, London June 10-11. 2014
Background

- Customary **Nash equilibrium** has **no coupling constraints**.

- **Here**: coupling constraints are crucial; **sharing** of input/output.

- **Exchange** is part of the game; driven by gradient differences.

- **Walras equilibrium** in exchange markets is part of the solution.

- Agent-based adjustments; no coordination.

- Myopic and adaptive behavior.
- **Still**: convergence to equilibrium.
Issues

- Typically, Nash equilibrium isn’t Pareto efficient.

- But, Walras equilibrium is! (the so-called welfare theorems). Hence only partial efficiency.

- Where do market prices come from?

- Is price-taking behavior meaningful?

- Personal price ↔ willingness to pay.

- Is full optimization & perfect foresight needed?
The Game

Player $i \in I$

- uses strategy $x_i \in X_i$ (a Euclidean space)

- gets payoff $\pi_i(x_i, x_{-i}) \in \mathbb{R} \cup \{-\infty\}$

- is subject to coupling constraints:

$$ (x_i) =: x \in X \subset X := \prod_{i \in I} X_i \quad \text{where} \quad X \not\subseteq \prod_{i \in I} P_i X. $$

- **Nash equilibrium** $x \iff$

$$ \pi_i(x_i, x_{-i}) = \max \{ \pi_i(\hat{x}_i, x_{-i}) : (\hat{x}_i, x_{-i}) \in X \} > -\infty \quad \forall i. $$

- **Normal (Nash) equilibrium** $x \iff$

$$ x \in \operatorname{arg\ max} \left\{ \sum_{i \in I} \pi_i(\hat{x}_i, x_{-i}) : \hat{x} \in X \right\} > -\infty. $$
On normal equilibrium

- **Existence** of normal equilibrium is guaranteed when \( X \) is non-empty compact convex, and

\[
\pi(\hat{x}, x) := \sum_{i \in I} \pi_i(\hat{x}_i, x_{-i})
\]

is concave in \( \hat{x} \), usc in \((\hat{x}, x)\), & continuous in \( x \).

- **Characterization** of normal equilibrium \( x \) comes in 3 equivalent ways:
  * There is a margin \( m \in M(x) \cap N(x) \) where

\[
M(x) := \left. \frac{\partial}{\partial \hat{x}} \pi(\hat{x}, x) \right|_{\hat{x}=x} \text{ and } N(x) := N(x, X) := \text{ normal cone to } X \text{ at } x \]

  * Each feasible deviation

\[
d \in D(x) := \mathbb{R}_+(X - x) \subseteq c/\mathbb{R}_+(X - x) =: T(x) = \text{ tangent cone}
\]

satisfies the variational inequality \( \langle m, d \rangle \leq 0 \).

  * The orthogonal projection of \( m \) onto the tangent cone is nil:

\[
0 = P_{T(x)}[m]
\]
Out-of-equilibrium adjustments

Is the differential inclusion

$$\dot{x} \in P_{T(x)} [M(x)], \quad x(0) \in X, \quad (1)$$

a reasonable model of disequilibrium play?

One appeal: It’s driven by local, decentralized data

$$M_i(x) := \frac{\partial}{\partial \hat{x}_i} \pi_i(\hat{x}_i, x_{-i}) \big|_{\hat{x}_i = x_i}$$

Two drawbacks: First, continuous time isn’t quite realistic. Second, it’s not decentralized!

However, (1) helps in identifying a stability condition: a normal equilibrium $\bar{x}$ is asymptotically stable - hence unique - under (1) if

$$x \in X \backslash \bar{x} \implies \langle M(x), x - \bar{x} \rangle < 0. \quad (2)$$

(2) holds when $x \in X \backslash \bar{x}$ implies

$$\pi(\bar{x}, x) > \pi(x, x) \quad \text{or} \quad \langle M(x) - M(\bar{x}), x - \bar{x} \rangle < 0.$$
Coupling constraints

Henceforth

\[ X = \left\{ x = (x_i) : x_i = (y_i, z_i) \in Y_i \times Z_i \text{ } \& \text{ } \sum z_i \leq \sum e_i \right\} . \]

\(Y_i, Z_i\) are non-empty compact convex \(\mathbb{C}\) Euclidean spaces \(\mathbb{Y}_i, \mathbb{Z}\).

- Agent \(i\) owns endowment \(e_i\).
- \(\pi_i(x_i, x_{-i}) = \pi_i(y_i, y_{-i}, z_i)\). Increasing in \(z_i\) for at least one \(i\).

A workhorse instance: **Cournot oligopoly** (1838): Firm \(i \in I\) produces \(y_i\), uses production factors \(z_i\), incurs cost \(c_i(y_i, z_i)\), and gets payoff

\[ \pi_i(y_i, y_{-i}, z_i) = P(y_I) \cdot y_i - c_i(y_i, z_i) \text{ where } y_I := \sum_{i \in I} y_i. \]
The solution concept

A feasible profile \( x = (x_i) \in X \) and a price vector \( p \in \mathbb{Z}^* \) is a **Nash-Walras equilibrium** iff

\[
x_i = (y_i, z_i) \text{ maximizes } \pi(\hat{y}_i, y_{-i}, \hat{z}_i) - \langle p, \hat{z}_i \rangle \text{ s. t. } (\hat{y}_i, \hat{z}_i) \in Y_i \times Z_i \quad \forall i,
\]

and all exchange markets clear:

\[
p \geq 0, \quad \sum_{i \in I} z_i \leq \sum_{i \in I} e_i, \quad \text{and} \quad \left\langle p, \sum_{i \in I} e_i - \sum_{i \in I} z_i \right\rangle = 0.
\]
Getting around projection

Recall: in
\[ \dot{x} \in P_{T(x)}[M(x)], \quad x(0) \in X, \]
projection does not square with decentralized play.

Idea: Since coupling constraints are fairly simple - in fact, additive - bilateral exchange might take care of projection.
Where does the price come from?

Recall that in maximizing

\[ \pi_i(\hat{y}_i, y_{-i}, \hat{z}_i) - \langle p, \hat{z}_i \rangle \quad \text{s. t.} \quad (\hat{y}_i, \hat{z}_i) \in Y_i \times Z_i \]

agent \( i \) must predict \( y_{-i} \) and \( p \).
- In particular, where did \( p \) come from?
- How could \( p \) be learned?

Idea: Again, bilateral exchanges become instrumental. These are likely to generate \( p \).
Repeated play under additive coupling

There are two time scales (or clocks).
The fast one secures that

\[ \pi_i(y_i, y_{-i}, z_i) = \max \pi_i(Y_i, y_{-i}, z_i) \quad \forall i \]

That is, at any time

\[ y \in \text{Nash}(z). \]

The slow one regulates updates of \( z = (z_i) \)
Repeated play unfolding as a discrete-time algorithm

- **Start** at some feasible allocation \( i \mapsto z_i \in Z_i \) of transferable goods.
- **Find contingent Nash equilibrium**

\[ y \in \text{Nash}(z). \]

- **Two agents** \( i, j \) meet with uniform probability, holding \( z_i \in Z_i \) and \( z_j \in Z_j \) respectively. **Update their holdings** so that

\[ z_i^{+1} := z_i + sd_{ij} \in Z_i \quad \text{and} \quad z_j^{+1} := z_j - sd_{ij} \in Z_j \]

with step size \( s \geq 0 \) and direction

\[ d_{ij} \in \left[ \frac{\partial}{\partial z_i} \pi_i(y, z_i) - N(z_i, Z_i) \right] - \left[ \frac{\partial}{\partial z_j} \pi_j(y, z_j) - N(z_j, Z_j) \right]. \]

- **Continue** to find contingent Nash equilibrium until convergence.
Convergence

**Theorem** Suppose repeated play, as modelled above, proceeds at discrete stages $k = 0, 1, \ldots$ with step sizes $s_k \geq 0$ that satisfy

\[
\sum_{k=0}^{\infty} s_k = +\infty \quad \text{and} \quad \sum_{k=0}^{\infty} s_k^2 < +\infty.
\]

Then, under asymptotic stability, the generated sequence $x^k, k = 0, 1, \ldots$ converges to the unique Nash-Walras equilibrium.
Upshot so far

- There are several agents,
- each holding his (stochastic) endowment = (random) resource bundle = contingent claim.
- They proceed (along Nash equilibrium) by repeated bilateral barters.

**Issues:**
- Will full Nash-Walras equilibrium obtain? Quit likely: **Yes**!
- Is there room for "simple" agents? **Yes**!
- Is full optimization & perfect foresight really needed? **No**!
- Must prices be announced? **No**!
In line with behavioral economics

- Agents lack computational competence, perfect foresight, information, global vision, ....

- There is no coordination, no auctioneer, no market maker,...

- Bargaining, matching, search is not made explicit - and not really needed.

- Nonetheless: strategies & holdings converge to equilibrium.

- Inspiration from Pareto: "The economy is a great computing machine."
Back to bilateral barters

- Agent $i$ meets agent $j$. Their actual "states" are $x_i = (y_i, z_i)$ and $x_j = (y_j, z_j)$.
- Fix $y$ and posit $\Pi_i(z_i) := \pi_i(y, z_i)$, $\Pi_j(z_j) := \pi_j(y, z_j)$.
- Recall that

$$z_i^{+1} := z_i + sd_{ij} \in Z_i \text{ and } z_j^{+1} := z_j - sd_{ij} \in Z_j$$

with step size $s \geq 0$ and direction

$$d_{ij} \in \left[ \frac{\partial}{\partial z_i} \Pi_i(z_i) - N(z_i, Z_i) \right] - \left[ \frac{\partial}{\partial z_j} \Pi_j(z_j) - N(z_j, Z_j) \right].$$

- What is the nature of (a good direction) $d_{ij}$?
Intermezzo: the nature of the Walrasian part of equilibrium

**Definition** A feasible allocation \((z_i)\) of \(e_I := \sum_{i \in I} e_i\) and a linear price \(p: Z \rightarrow \mathbb{R}\) constitute a **Walras equilibrium** iff

\[
\Pi_i(z_i) + p(e_i - z_i) \geq \Pi_i(\hat{z}_i) + p(e_i - \hat{z}_i) \quad \text{for each } i \in I \text{ and } \hat{z}_i \in Z_i.
\]

In equilibrium each agent maximizes his payoff + his net value of sales.

**Proposition (On Walras equilibrium)** A profile \((z_i)\) alongside a price \(p\) constitutes a Walras equilibrium iff

\[
p \in \partial \Pi_i(z_i) - N_i(z_i) \quad \text{for each } i, \quad \text{and} \quad \sum_{i \in I} z_i = e_I. \quad \square
\]

- that is: there is a *common* price
- price "=" marginal utility (modulo normal components)
More on the nature of equilibrium: Cooperative aspects

- Suppose coalition $C \subseteq I$ uses aggregate endowment $e_C := \sum_{i \in C} e_i$ to get (sup-convolution)
  \[ \Pi_C(e_C) := \sup \left\{ \sum_{i \in C} \Pi_i(z_i) : \sum_{i \in C} z_i = e_C \land z_i \in Z_i \right\}. \]
- Payment scheme $(\pi_i)$ is in the core of this transferable-utility game iff
  \[
  \begin{cases}
  \text{Pareto efficient:} & \sum_{i \in I} \pi_i = \Pi_I(e_I) \\
  \text{stable against blocking:} & \sum_{i \in C} \pi_i \geq \Pi_C(e_C) \text{ for each } C \subseteq I.
  \end{cases}
  \]

**Proposition (Equilibrium as core solution)** For any equilibrium price $p$ the payment scheme

\[
i \mapsto \pi_i := \sup \left\{ \Pi_i(\hat{z}_i) + p(e_i - \hat{z}_i) : \hat{z}_i \in Z_i \right\}
\]

is in the core. □
Maintaining feasibility throughout

Cone of feasible directions for agent $i$ at $z_i$:

$$D_i(z_i) := \mathbb{R}_+(Z_i - z_i).$$

Cone of common directions

$$D_{ij}(z_i, z_j) := D_i(z_i) \cap -D_j(z_j).$$

Tangent cone

$$T_{ij}(z_i, z_j) := \text{cl}D_{ij}(z_i, z_j).$$

Maximal slope of joint improvement

$$\mathcal{S}_{ij}(z_i, z_j) := \max_d \left\{ \Pi_i'(z_i; d) + \Pi_j'(z_j; -d) : d \in T_{ij}(z_i, z_j) \text{ and } \|d\| \leq 1 \right\}.$$
On the maximal slope

\[ \mathcal{S}_{ij}(z_i, z_j) := \max_d \{ \Pi'_i(z_i; d) + \Pi'_j(z_j; -d) : d \in T_{ij}(z_i, z_j) \land \|d\| \leq 1 \}. \]

**Proposition** Let \( P_{ij} = \) orthogonal projection onto \( T_{ij}(z_i, z_j) \):

\[
\mathcal{S}_{ij}(z_i, z_j) = \min \{ \| P_{ij} [g_i - g_j]\| : g_i \in \partial \Pi_i(z_i), \ g_j \in \partial \Pi_j(z_j) \}
\]

\[
= \text{dist} [\partial \Pi_i(z_i) - N_i(z_i), \partial \Pi_j(z_j) - N_j(z_j)],
\]

where \( N_i(z_i) \) is the normal cone to \( Z_i \) at \( z_i \).
Material transfers

- Fix an Armijo parameter \( \varphi_{ij} \in (0, 1) \) for each agent pair \( i, j \).

**Definition:** While holding \( z_i \in Z_i \) and \( z_j \in Z_j \), agents \( i, j \), make a real transfer \( sd \), with \( s > 0 \) and \( \|d\| = 1 \), if

\[
    z_i + sd \in Z_i \quad \text{and} \quad z_j - sd \in Z_j
\]

and

\[
    \Delta \Pi_{ij} := \Pi_i(z_i + sd) + \Pi_j(z_j - sd) - \Pi_i(z_i) - \Pi_j(z_j)
\]

satisfies

\[
    \Delta \Pi_{ij} \geq s \varphi_{ij} \mathcal{S}_{ij}(z_i, z_j).
\]

**Proposition** When \( \mathcal{S}_{ij}(z_i, z_j) > 0 \), there exists a real transfer.
\[ \Delta \Pi_{ij} > 0 \iff \exists \text{ side payments } \Delta r_i \text{ and } \Delta r_j \text{ such that } \Delta r_i + \Delta r_j = 0 \text{ and} \]

\[ \Pi_i(z_i^{+1}) + \Delta r_i > \Pi_i(z_i) \quad \& \quad \Pi_j(z_j^{+1}) + \Delta r_j > \Pi_j(z_j) \]

is solvable.

- Money "oils" the transaction machinery.

- Deals and incentives are compatible.
Recall that in equilibrium

\[ p \in \cap_{i \in I} [\partial \Pi_i(z_i) - N_i(z_i)] . \]

We say agents \( i, j \) see a common price iff

\[ [\partial \Pi_i(z_i) - N_i(z_i)] \cap [\partial \Pi_j(z_j) - N_j(z_j)] \neq \emptyset . \]

**Proposition** \( i, j \) see a common price iff

\[ \mathcal{S}_{ij}(z_i, z_j) = 0. \]

Thus improvement is possible as long as some \( \mathcal{S}_{ij}(z_i, z_j) > 0 \).
Asymmetric information

\[ Z = L^0(S, \mathcal{F}, \mu, \mathbb{E}) \text{ with } \mathcal{F} \text{ finite.} \]

\[ Z_i = L^0(S, \mathcal{F}_i, \pi, \mathbb{E}) \text{ with } \mathcal{F}_i \subseteq \mathcal{F}. \]

\[ z_i \in Z_i \implies D_i(z_i) = Z_i. \]

and \( d \in T_{ij}(z_i, z_j) = Z_i \cap Z_j \implies d \text{ constant on } A_i \cup A_j \text{ when atoms } A_i \in \mathcal{F}_i \text{ and } A_j \in \mathcal{F}_j \text{ intersect.} \]

Projection = conditional expectation:

\[ \Pr(z)_s = \frac{\sum_{s \in A} z_s \mu_s}{\sum_{s \in A} \mu_s} \text{ for each } s \in \text{atom } A. \]
Concluding remarks on equilibrium and dynamics?

- How can players arrive at equilibrium - if any?

- While underway, how much competence, coordination, and foresight is required?

- What are the roles of cognition and perception?
• It requires no coordination, experience, foresight, or optimization.

• It’s totally decentralized.

• It’s fully driven by low-complexity adaptive agents.
References

Flåm & Gramstad, Direct exchanges in linear economies, *Int. J. Game Th* (2013)
Shapley & Shubik, Trade using one commodity as a means of payment, *J Pol Econ* (1977)