VARIATIONAL INEQUALITY MODELING OF EQUILIBRIUM IN FINANCIAL MARKETS

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Economic Equilibrium Theory — Overview

Modeling territory:
situations where competing tendencies must be balanced
such as games with multiple optimizers, but going beyond

Role in the theory of markets:
coordinating agents’ subproblems of utility maximization
making demands adjust to meet supplies
understanding how prices can decentralize decision-making

Mathematical issues:
• existence? under assumptions appropriate for the situation
• computation? passing from qualititave to quantitative
• stability? related also to “local uniqueness”!

Methodologies to employ:
fixed point theory, optimization theory, convex analysis
variational analysis, variational inequalities, . . .
agents $i = 1, \ldots, m$, goods vectors $x \in \mathbb{R}^n_+$, price vectors $p \in \mathbb{R}^n_+$

Optimization subproblems for the agents:
- agent $i$ starts with a goods vector $x^0_i \in \mathbb{R}^n_+$
- trades it for a goods vector $x_i \in \mathbb{R}^n_+$ in a market with prices $p$
- must respect the budget constraint $p \cdot x_i \leq p \cdot x^0_i$
- seeks to maximize its associated utility $u_i(x_i)$

Equilibrium problem (parameterized by initial holdings $x^0_i$)

Determine $\bar{x}_i$ for $i = 1, \ldots, m$ and a price vector $\bar{p}$ such that

(a) $\bar{x}_i$ maximizes $u_i(x_i)$ subject to the budget dictated by $\bar{p}$,

(b) $\sum_{i=1}^m x^0_i - \sum_{i=1}^m \bar{x}_i \geq 0$, $\bar{p} \cdot \left[ \sum_{i=1}^m x^0_i - \sum_{i=1}^m \bar{x}_i \right] = 0$

supply-demand conditions with complementary slackness
Issues in the Basic One-Stage Model

Modeling shortcomings:
- isolation in time with no future planning; doomsday effects
- no economic mechanism for the determination of prices

Existence shortcomings:
- reliance on nonconstructive fixed-point approaches
- inadequate structure for computational developments

Conclusion shortcomings:
- typically all components of initial $x_i^0$ must be $> 0$
- stability, entailing also local uniqueness, seems elusive
- widespread fallback on results that are merely generic

Recent progress:
- assumptions on initial holdings greatly weakened
- stability shown to prevail far more than anticipated
- achieved through variational inequality methodology
- utility functions concave, not just quasi-concave
Variational Inequality Framework — With Normal Cones

Variational inequality: $-f(z) \in N_C(z) \quad C = \text{closed convex set}$

Composite modeling: $-f_k(z_1, \ldots, z_r) \in N_{C_k}(z_k)$ for $k = 1, \ldots, r$

corresponds to $-f(z_1, \ldots, z_r) \in N_C(z_1, \ldots, z_r)$ when

$$C = C_1 \times \cdots \times C_r,$$

$$f(z_1, \ldots, z_r) = (f_1(z_1, \ldots, z_r), \ldots, f_r(z_1, \ldots, z_r))$$

Complementary slackness conditions as an example:

cases where $C_k = \text{some orthant } R^n_+$ or even just $R_+$

Applications to economic equilibrium when utility is concave

- utility maximization characterized by saddle point conditions
- saddle points have variational inequality representations
- truncation arguments facilitated by appeals to duality

Incompleteness of current utility theory?

it only gets quasi-concavity by neglecting “marginal utility”?
Variational Inequality Representation of the Basic Model

• assume that all the utility functions are $C^1$ and concave
• introduce multipliers $\lambda_i$ for the agents’ budget constraints

Lagrangians for utility maximization:

$$L_i(x_i, \lambda_i) = u_i(x_i) - \lambda_i p \cdot [x_i - x_i^0] \text{ for } x_i \in \mathbb{R}^n_+, \lambda_i \in \mathbb{R}_+$$

Conditions for a variational inequality in $z = (p, \ldots, x_i, \lambda_i, \ldots)$

(a) $\nabla u_i(x_i) - \lambda_i p \in N_{\mathbb{R}^n_+}(x_i) \text{ for } i = 1, \ldots, m$
(b) $p \cdot [x_i - x_i^0] \in N_{\mathbb{R}_+^m}(\lambda_i) \text{ for } i = 1, \ldots, m$
(c) $\sum_{i=1}^m x_i - \sum_{i=1}^m x_i^0 \in N_{\mathbb{R}^n_+}(p)$

(1) this V.I. generally won’t be of monotone type
(2) this V.I. has form $-f(z) \in N_C(z)$ with $C$ unbounded

Existence and stability: available under agreeable assumptions through the innovation of introducing money as a “good”
Economic role and motivation:
- saving for the future, borrowing from the future
- hedging against various possible scenarios, “insurance”

Multistage models: 1970’s, 1980’s, 1990’s
- discretization of time and uncertainty in future states
- real contracts to deliver/receive future “goods”
- nominal contracts to deliver/receive future “value” with “value” denominated in so-called “units of account”

Essential incompleteness of markets: beyond Arrow-Debreu
- available contract configurations are unable to hedge fully
- planning can’t be exact, even for the modeled future

→ GEI = general equilibrium theory with incomplete markets
Drawbacks of Current Versions of GEI

Troubles with establishing existence/uniqueness:

- equilibrium is problematical using real contracts
  game-changing counterexample of Hart 1976
  technical difficulties with keeping markets in check
- equilibrium is “indeterminate” using nominal contracts
  unscaled prices prevent comparisons between states

Counterintuitive features for a financial model:

- money is absent! — no exchange rates, inflation/deflation
- only “immediate consumption” of goods has “utility”
- doomsday effects of time horizon distort agent behavior
- no place for the “unmodeled uncertainty” of Keynes
- no coverage of “derivatives” or pre-existing “assets”
- agents are supposed to predict future prices correctly
A New and Different Approach

away from the limitations of just commodities and consumption

Goods from a much broader perspective

A “good” (generalized) may be anything that
- can freely be traded between agents
- is fixed in supply in any state, present or future

The “goods” possessed by an agent can, in any state,
- either be consumed or retained
- an agent’s utility balances consumption with retention
- retained goods pass (modified?) from present to future

Money as a special “good”:
agents always like to retain it and they can freely save it
justification from arguments of Keynes about uncertainty
⇒ money is able to serve in denominated all prices
More Detail on Goods, States and Utility

Goods: \( l = 0, 1, \ldots, L \), goods vectors \( \in \mathbb{R}_{1+L} \), money = good 0

States: \( s = 0 \) at time 0, \( s = 1, \ldots, S \) at time 1

Agents: \( i = 1, \ldots, I \) deal with goods vectors in all states \( s \),
getting \( e_i(s) \), retaining \( w_i(s) \), consuming \( c_i(s) \)

Utility: \( u_i(w_i(0), \ldots, w_i(S); c_i(0), \ldots, c_i(S)) \) for agent \( i \)

Survival: \( (w_i(0), \ldots, w_i(S); c_i(0), \ldots, c_i(S)) = (w_i, c_i) \in U_i \)
\( u_i \) nondecreasing, \textbf{concave}, insatiable for retaining money

Passing to the future
\( w_i(0) \) in state \( s = 0 \) emerges as \( A_i(s)w_i(0) \) in states \( s > 0 \)

\[ A_i(s) \in \mathbb{R}_{1+L}^{(1+L) \times (1+L)} \] for \( s = 1, \ldots, S \)
the free saving of money is incorporated into this
Two-Party Contracts as Financial Instruments

with money as a “good,” only “real” contracts are needed!

Contract types: \( k = 0, 1, \ldots, K \); contract 0 will be special
- the goods vector \( D_k(s) \) is delivered in each state \( s > 0 \),
- the goods vector \( D_k(0) \) is consumed in the state \( s = 0 \)
the latter will induce “transaction costs” endogenously

Contract markets:
- the contracts can be bought and sold by the agents
  the purchaser gets the future deliveries from the seller,
  the seller provides for the required initial consumption
- fractional amounts allowed, no limit on quantities

Lending and borrowing money: contract 0
this delivers a unit of good 0 in every state \( s > 0 \)
purchaser = money lender, seller = money borrower
Market Prices and Optimization

all prices are denominated in units of good 0, money

**Prices of goods:**  \( p_l(s) \geq 0 \) for good \( l \) in states \( s = 0, 1, \ldots, S \)

**Prices of contracts:**  \( q_k \geq 0 \) for contract \( k \) in state \( s = 0 \)

\[
p(s) = (1, p_1(s), \ldots, p_L(s)), \quad q = (q_0, q_1, \ldots, q_K)
\]

Contract activity of the agents

agent \( i \) buys \( z_i^+ = (z_{i1}, \ldots, z_{iK}) \) and sells \( z_i^- = (z_{i1}, \ldots, z_{iK}) \)

Budget constraints:  in present and future states

\[
p(0)[w_i(0) + c_i(0) - e_i(0) + D(0)z_i^+] + q[z_i^+ - z_i^-] \leq 0,
\]

\[
p(s)[w_i(s) + c_i(s) - e_i(0) - D(s)[z_i^+ - z_i^-] - A_i(s)w_i(0)] \leq 0
\]

\( D(s) = \) the matrix with delivery columns \( D_k(s) \)

Agent’s utility optimization problem

choose \( w_i(s), c_i(s), z_i^+, z_i^- \) to maximize utility \( u_i(w_i, c_i) \)

subject to survival \( (w_i, c_i) \in U_i \) and the budget constraints
Main Result for the Financial Model

**Definition of equilibrium in prices and decisions**

1. the agents’ choices solve their utility problems
2. excess demands for goods are \( \leq 0 \), but \( = 0 \) if price > 0
3. total contracts bought = total contracts sold

**Ample survivability assumption:** remarkably weak

- even without entering markets the agents can “survive” while
  - individually not exhausting all their money at time 0
  - collectively leaving a surplus of other goods \( l \) in all states \( s \)

**Existence of equilibrium**

Under the ample survivability assumption, the utility assumptions and some others (minor), an equilibrium is guaranteed to exist

**Advanced V.I. framework:** normal cones \( \rightarrow \) subdifferentials

\( \exists \) ongoing research on stability/“local uniqueness” of equilibrium
Some References

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