Optimal insurance with counterparty default risk

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December 7, 2010
Agenda
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Reinsurance purchases

Source: Guy Carpenter & Co., see Froot (1997, 2001)
## CDS Spreads

**TRX P&C Re Index**

<table>
<thead>
<tr>
<th>Company</th>
<th>Net Premiums (USD)</th>
<th>1Y CDS Spread</th>
<th>Weight</th>
<th>Index contrib</th>
</tr>
</thead>
<tbody>
<tr>
<td>Munich Re</td>
<td>28,384m</td>
<td>26.26</td>
<td>28.92%</td>
<td>7.59</td>
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<tr>
<td>Swiss Re</td>
<td>23,770m</td>
<td>42.20</td>
<td>24.22%</td>
<td>10.22</td>
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<tr>
<td>Berkshire Hathaway</td>
<td>10,650m</td>
<td>107.36</td>
<td>10.85%</td>
<td>11.65</td>
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<tr>
<td>Hannover Re</td>
<td>10,640m</td>
<td>37.22</td>
<td>10.84%</td>
<td>4.03</td>
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<tr>
<td>Lloyd’s of London</td>
<td>8,593m</td>
<td>145.86</td>
<td>8.76%</td>
<td>12.78</td>
</tr>
<tr>
<td>SCOR</td>
<td>7,826m</td>
<td>39.21</td>
<td>7.97%</td>
<td>3.13</td>
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<tr>
<td>Everest Re</td>
<td>3,505m</td>
<td>45.22</td>
<td>3.57%</td>
<td>1.61</td>
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<tr>
<td>XL Capital</td>
<td>2,402m</td>
<td>55.81</td>
<td>2.45%</td>
<td>1.37</td>
</tr>
<tr>
<td>Renaissance Re</td>
<td>1,354m</td>
<td>95.02</td>
<td>1.38%</td>
<td>1.31</td>
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<tr>
<td>Ace</td>
<td>1,019m</td>
<td>38.12</td>
<td>1.04%</td>
<td>0.40</td>
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**Index value** 54.09

Contagion

TRX P&C Re Index, April 2008 - January 2010

Source: Thomson Reuters.
Counterparty default risk

Limited risk sharing of large (catastrophic) risks

Insurance cycles
- Cummins/Danzon (1997): insolvency risk, no contract design

Optimal insurance contracts (Borch, Arrow, Raviv, Gollier, etc.)
- Tapiero/Kahane/Jaques (1986): coinsurance, endogenous default
- Cummins/Mahul (2003): heterogeneous beliefs, exogenous default
- Richter (2004): hedging (basis risk) vs. insurance (exogenous default)
- Dana/Scarsini (2007): background risk
Setup

One-period, no discounting.

Insurance buyer

- utility $u(\cdot)$ ($u' > 0, u'' < 0$), random wealth $W \geq 0$
- insurable risk $X$ valued in $[0, \bar{x}]$
- premium $P \geq 0$, indemnity $I(x)$ on $\{X = x\}$

$$W - X - P + I(X)$$

Insurer

- risk-neutral, random assets $A \geq 0$
- receives premium $P$, pays $I(X)$

$$A + P - I(X)$$
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Insurer

- risk-neutral, random assets $A \geq 0$
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$$\max\{A + P - I(X), 0\}$$
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One-period, no discounting.

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- premium $P \geq 0$, indemnity $I(x)$ on $\{X = x\}$

\[
(W - X - P + I(X)) 1_{\{A + P - I(X) \geq 0\}} + \\
(W - X - P + (A + P) \gamma) 1_{\{A + P - I(X) < 0\}}, \quad (0 \leq \gamma \leq 1)
\]

Insurer

- risk-neutral, random assets $A \geq 0$
- receives premium $P$, pays $I(X)$

\[
\max\{A + P - I(X), 0\}
\]
Setup

One-period, no discounting.

Insurance buyer

- utility $u(\cdot)$ ($u' > 0, u'' < 0$), random wealth $W \geq 0$
- insurable risk $X$ valued in $[0, \bar{x}]$
- premium $P \geq 0$, indemnity $I(x)$ on \{ $X = x$ \}

$$\tilde{W} := W - X - P + I(X)$$

$$\tilde{W}(\gamma) := W - X - P + (A + P) \gamma$$

Insurer

- risk-neutral, random assets $A \geq 0$
- receives premium $P$, pays $I(X)$

$$\tilde{A} := A + P - I(X)$$
Agenda
Efficient insurance contracts

\[
U(P, I) := E \left( 1_{\tilde{A} \geq 0} u(\tilde{W}) + 1_{\tilde{A} < 0} u(\tilde{W}(\gamma)) \right)
\]

\[
V(P, I) := E(\max\{\tilde{A}, 0\})
\]
Efficient insurance contracts

\[ U(P, I) := E \left( 1_{A \geq 0} u(\tilde{W}) + 1_{A < 0} u(\tilde{W}(\gamma)) \right) \]
\[ V(P, I) := E(\max\{\tilde{A}, 0\}) \]

Find efficient \((P^*, I^*)\) by solving the following problem for different values of insurer’s reservation utility \(v\)

\[
\begin{align*}
\sup_{(P, I) \in \mathbb{R}_+ \times \mathcal{A}} & \quad U(P, I) \\
\text{subj. to} & \quad V(P, I) \geq v \\
& \quad 0 \leq I \leq Id
\end{align*}
\]

Assumptions

- \(A\) piecewise continuous functions
- \(A, W \in L^1\) (problem is well-posed)
- \((A, W, X)\) has continuous positive density \(f\) on \(\mathbb{R}^2_+ \times [0, \bar{x}]\)
Efficient insurance contracts

\[
U(P, I) := E \left( 1_{\tilde{A} \geq 0} u(\tilde{W}) + 1_{\tilde{A} < 0} u(\tilde{W}(\gamma)) \right)
\]

\[
V(P, I) := E(\max\{\tilde{A}, 0\})
\]

**First step:** for fixed \( P \geq 0 \) and \( v \in [E(A), E(A) + P] \), solve

\[
\left\{ \begin{array}{l}
\sup_{I \in A} U(P, I) \\
\text{subj. to} \quad \int_0^x E(\max\{A + P - I(x), 0\} | X = x) f_X(x) dx \geq v \\
\quad 0 \leq I \leq Id
\end{array} \right.
\]

**Second step:** optimize over \( P \).

**Participation constraints**

- insurer: \( v \geq E(A) = V(0, 0) \)
- insurance buyer: \( U(P^*, I^*) \geq E(u(W - X)) = U(0, 0) \)
Optimal insurance for given $P$

Two key functions

$$
\overline{J}(x) := u'(w - P - x) - \lambda
$$

$$
\underline{J}(x) := u'(w - P) - \lambda
$$

The zeros of $\overline{J}$, $\underline{J}$ (say $\{x_i\}, \{x_j\}$) determine changes in coverage level

- in traditional settings, $\overline{J}$ and $\underline{J}$ are monotone: there is at most a positive zero (deductible, $x_1$, or upper limit, $x_1$)
Optimal insurance for given $P$

Two key functions

$$\overline{J}(x) := E(u'(W - P - x) - \lambda | X = x)$$

$$\underline{J}(x) := E((u'(W - P) - \lambda) 1_{A \geq x - P} | X = x)$$

$$- f_A(x - P | x) E(\Delta u(\gamma, I) | \tilde{A} = 0, X = x)$$

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Crucial in our setting

- law of $(A, W, X)$

- expected utility loss when crossing the default boundary, $\tilde{A} = 0$
Optimal insurance for given $P$

Two key functions

$$
\bar{J}(x) := E(u'(W - P - x) - \lambda | X = x)
$$

$$
\underline{J}(x) := E\left((u'(W - P) - \lambda) 1_{A \geq x - P} | X = x\right) - f_A(x - P|x) E(\Delta u(\gamma, I)|\tilde{A} = 0, X = x)
$$

The zeros of $\bar{J}, \underline{J}$ (say $\{\bar{x}_i\}, \{\underline{x}_j\}$) determine changes in coverage level

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Crucial in our setting

• law of $(A, W, X)$

• expected utility loss when crossing the default boundary, $\tilde{A} = 0$

$$
\Delta u(\gamma, I) = u(\tilde{W}) - u(\tilde{W}(\gamma))
$$
Optimal insurance for given $P$

Two key functions

$$
\overline{J}(x) := E \left( u'(W - P - x) - \lambda | X = x \right)
$$

$$
\underline{J}(x) := E \left( (u'(W - P) - \lambda) 1_{A \geq x - P} | X = x \right)
$$

$$
- f_A(x - P | x) E(\Delta u(\gamma, I) | \tilde{A} = 0, X = x)
$$

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Crucial in our setting

- law of $(A, W, X)$
- expected utility loss when crossing the default boundary, $\tilde{A} = 0$

$$
\lim_{a \uparrow 0} E(\Delta u(\gamma, I) | \tilde{A} = a, X = x) = E(\Delta u(\gamma, I) | \tilde{A} = 0, X = x)
$$
Traditional case
Traditional case

![Graph showing two curves labeled $\overline{J}$ and $J$ against $x$ and $\overline{x}$]
Traditional case

\[ \overline{J}, \quad J \]

\[ x, \quad \overline{x} \]

0
Traditional case

\[ I^*(x) \]

\[ \overline{J} \]

\[ \underline{J} \]

\[ 0 \]

\[ x \]

\[ \overline{x} \]
Traditional case

\[ \frac{I^*(x)}{x} \]

Graph showing the function \( I^*(x)/x \) with curves \( \overline{J} \) and \( \underline{J} \) and the x-axis labeled \( x \) and the y-axis labeled \( \overline{x} \).
Traditional case (regulatory constraint on $P$)

\[ 0 < I^*(x) < x \]
Traditional case (regulatory constraint on $P$)
Traditional case (regulatory constraint on $P$)
Traditional case (claims handling costs)

\[ I^*(x) = 0 \]

\[ 0 < I^*(x) < x \]
Traditional case (claims handling costs)
Traditional case (claims handling costs)

\[ \frac{I^*(x)}{x} \]

\[ J \]

\[ x \]

\[ \bar{x} \]
Our setting
Our setting
Our setting
Our setting

$I^*(x)$

$\overline{J}$

$\overline{\overline{J}}$

$0$

$x$

$\overline{x}$

N

C

F

C
Our setting

\[ I^*(x)/x \]
Agenda
Main insights

Efficient contracts I

• For given $v \geq E(A)$ and optimal $(P^*, I^*)$

$$P^* = E(I^*(X)) + (v - E(A)) - E(\max\{I^*(X) - (A + P^*), 0\}),$$

and $(P^*, I^*)$ is efficient.

• If no insurance efficient, then $E(u'(W - x) | X = x)$ constant in $x$.

• With bankruptcy costs ($0 \leq \gamma < 1$), any efficient contract provides no insurance on a set of positive measure.
Main insights

Efficient contracts I

- For given $v \geq E(A)$ and optimal $(P^*, I^*)$

$$P^* = E(I^*(X)) + (v - E(A)) - E(\max\{I^*(X) - (A + P^*), 0\}),$$

and $(P^*, I^*)$ is efficient.

- If no insurance efficient, then $E(u'(W - x)|X = x)$ constant in $x$.

- With bankruptcy costs ($0 \leq \gamma < 1$), any efficient contract provides no insurance on a set of positive measure.

Implications

- deductibles without administrative costs (Raviv, 1979) or background risk (Dana/Scarsini, 2007)

- upper limits without regulatory constraints (Raviv, 1979; Jouini/Schachermayer/Touzi, 2008) or policyholder’s limited liability (Huberman/Mayers/Smith, 1983)
Coinsurance rates

**Efficient contracts II**

When $I^*$ differentiable and interior, it satisfies

$$I^*(x) = \frac{\delta_0(x) - \delta_1(x) - h(x)(\delta_5(x, \gamma) - \delta_6(x, \gamma) + \delta_7(x, \gamma))}{\delta_0(x) - h(x)(\delta_2(x) + \delta_3(x, \gamma) + \delta_4(x, \gamma))}$$

- hazard rate $h(x) := f_A(I^*(x) - P|x)/\mathbb{P}(\tilde{A}^* \geq 0|X = x)$
- $\delta$’s admit explicit expressions dependent on $\gamma$ & law of $(A, W, X)$: e.g., $\delta_1(x) = \text{Cov}(u'(\tilde{W}^*), \ln_x f(A, W|x)|\tilde{A}^* \geq 0, X = x)$
**Coinsurance rates**

**Efficient contracts II**

When \( I^\ast \) differentiable and interior, it satisfies

\[
I^\ast'(x) = \frac{\delta_0(x) - \delta_1(x) - h(x)(\delta_5(x, \gamma) - \delta_6(x, \gamma) + \delta_7(x, \gamma))}{\delta_0(x) - h(x)(\delta_2(x) + \delta_3(x, \gamma) + \delta_4(x, \gamma))}
\]

- hazard rate \( h(x) := f_A(I^\ast(x) - P|x)/\mathbb{P}(\tilde{A}^\ast \geq 0|X = x) \)
- \( \delta \)'s admit explicit expressions dependent on \( \gamma \) & law of \((A, W, X)\):
  - e.g., \( \delta_1(x) = \text{Cov}(u'(\tilde{W}^\ast), \ln x f(A, W|x)|\tilde{A}^\ast \geq 0, X = x) \)

**Implications**

- insured fraction may be tent-shaped
- nonmonotonicity results of Schlesinger/vonSchulenburg (1987), Doherty/Schlesinger (1990) far from surprising
- background risk and default risk *jointly* shape coinsurance rates

\[
I^\ast'(x) = \frac{\delta_0(x) - h(x)\delta_5(x, \gamma)}{\delta_0(x) - h(x)(\delta_3(x, \gamma) + \delta_4(x, \gamma))}
\]

when \( A, W, X \) independent
Concepts of positive dependence

Stochastic increasingness

- \( Y \uparrow_{st} Z \) if conditional distribution of \( Y \), given \( Z \), becomes larger (FOSD) when conditioning to larger values of \( Z \)

\[
E(g(Y)|Z = z) \uparrow z \text{ for all } g \text{ nondecreasing}
\]

- example: \( W \uparrow_{st} -X \) (i.e., \( W \downarrow_{st} X \)), policyholder’s wealth negatively affected by the insurable loss.
Concepts of positive dependence

Stochastic increasingness

- $Y \uparrow_{st} Z$ if conditional distribution of $Y$, given $Z$, becomes larger (FOSD) when conditioning to larger values of $Z$

$$E(g(Y)|Z = z) \uparrow z \; \text{for all } g \text{ nondecreasing}$$

- example: $W \uparrow_{st} -X$ (i.e., $W \downarrow_{st} X$), policyholder’s wealth negatively affected by the insurable loss.

Affiliation $\Rightarrow$ stochastic increasingness

- $f_Y$ log-supermodular (Milgrom and Weber, 1982)

$$f_Y(y_1 \lor y_2)f_Y(y_1 \land y_2) \geq f_Y(y_1)f_Y(y_2) \quad \text{for all } y_1, y_2 \in \mathbb{R}^n$$

- examples

  $\star$ $(A, X)$ affiliated: insurer holds (say) industry loss warranties
  $\star$ $(A, W, -X)$ affiliated: loss negatively affects insurer/insured

- $\ln_i f(y_{-i}|y_i) := \frac{\partial}{\partial y_i} \ln f(y_{-i}|y_i)$ nondecreasing
Negative dependence

No bankruptcy costs

Assume $A \perp_X W$, $W \downarrow_{st} X$ and $\gamma = 1$. Then any efficient contract is either full insurance or a generalized deductible followed by coinsurance and full insurance.

- If $(W, -X)$ affiliated, $Id - I^*$ nonincreasing when $I^*$ interior.
- If $W \perp X$, full insurance efficient.
# Negative dependence

## No bankruptcy costs

Assume $A \perp_{X} W$, $W \downarrow_{st} X$ and $\gamma = 1$. Then any efficient contract is either full insurance or a generalized deductible followed by coinsurance and full insurance.

- If $(W, -X)$ affiliated, $Id - I^*$ nonincreasing when $I^*$ interior.
- If $W \perp X$, full insurance efficient.

## Bankruptcy costs

Assume $A \perp_{X} W$, $W \downarrow_{st} X$, and $0 \leq \gamma < 1$. Then any efficient contract involves a **positive deductible**.

- If $W \perp X$, the deductible is followed by coinsurance.
- If $W \perp X$, $(A, -X)$ affiliated, $\gamma = 0$, then $Id - I^*$ nondecreasing when $I^*$ interior.
Positive dependence

**No bankruptcy costs**

Assume $A \perp_{X} W$, $W - X \uparrow^{sst} X$, and $\gamma = 1$. Then any efficient contract involves full insurance, followed by coinsurance and no insurance.

- If $(W, X)$ affiliated, then $Id - I^{*}$ nondecreasing when $I^{*}$ interior.

**Bankruptcy costs**

Assume $A \perp_{X} W$, $W \uparrow^{sst} X$, and $0 \leq \gamma < 1$. Then any efficient contract involves an upper limit on coverage.
Sufficiency

Theorem

\((\hat{P}, \hat{I})\) admissible contract, \(\hat{\lambda} \geq 0\) corresponding adjoint constant. Assume \(A \perp_X W\) and that either of the following conditions hold:

- \(\gamma = 1\)
- \(\gamma = 0\) and \(\ln f_A(\hat{I}(x) - \hat{P}|x) \geq h(x)\).

Then, \((\hat{P}, \hat{I})\) is optimal if and only if the Maximum Principle applies, i.e.,

\[
H(x, \hat{P}, \hat{I}(x), \hat{\lambda}) = \max_{z \in [0, x]} H(x, \hat{P}, z, \hat{\lambda}), \quad \text{for all } x \in [0, \bar{x}]
\]

Existence

- the above allows us to prove existence of solutions case by case;
- procedure: identify switching points, ensure existence of relevant solutions to ODE on \(X_1\), verify that conditions above and of the Maximum Principle are satisfied.
$I^*(x)$ and $I^*(x)/x$: some examples

Assumptions

- $u(w) = -\exp(-\alpha w)$
- $A, W, X$ independent
- $W \sim \text{logNormal}(150, 35)$
- $A \sim \text{logNormal}(300, \sigma_A)$

Baseline parameters

- $\alpha = 0.05$
- $\sigma_A = 50$
- $\gamma = 0$ (total default)
Partial recovery and indemnity schedule

\[ I^*(x) \]

\[ \gamma = 0, \gamma = 0.25, \gamma = 0.50, \gamma = 0.75, \gamma = 1 \]
Partial recovery and insured fraction

\[ I^*(x)/x \]

\[ \gamma = 0 \]
\[ \gamma = 0.25 \]
\[ \gamma = 0.50 \]
\[ \gamma = 0.75 \]
\[ \gamma = 1 \]
Default probability

Exposure (x)

Fraction reinsured, \( \frac{I(x)}{x} \)

\( \sigma_A = 50 \)
\( \sigma_A = 40 \)
\( \sigma_A = 30 \)
\( \sigma_A = 20 \)
\( \sigma_A = 10 \)
Risk aversion

Exposure (x)

Fraction reinsured, I*(x) / x

α = 0.05
α = 0.16
α = 0.26
α = 0.38
α = 0.50
Agenda
Conclusion

Main findings

- contract design with endogenous counterparty risk
- explicit characterization of coinsurance rates
- role of dependence, limited liability, bankruptcy costs
- Pareto optimal insurance contracts
- necessary and sufficient conditions, existence

Positive and normative implications

- risk sharing patterns for high layers of exposures
- insolvency risk and insurance demand along the underwriting cycle
- optimal design of (re)insurance programs and government guarantees
Thank you